Useful Mathematical Formulas for Transform Limited Pulses

Aaron Webster

Last Update: December 3, 2012

1 Introduction

On the subject of pulses, there are many different mathematical formalisms which can be found in the literature. I've found all of them quite confusing and inadequate in clearly describing their properties. The purpose of this document, as the title suggests, is to present a useful mathematical formalism for representing pulses. It was written with ultrashort pulses of light in mind, but could easily be applied to any type of complex envelope function.

Formulas are presented for three different pulse types: the Gaussian, the sech, and the Lorentzian. They are expressed such that

- The total integrated pulse energy is unity.
- The full width at half max (FWHM) is a obvious variable in the pulse expression.
- They are conveniently expressed in either the time or frequency domain.

Furthermore, for each pulse type, analytic formulas for the time-bandwidth product and the total integrated energy with bounds are given.

1.1 Notation

In this document, the variables are used with the following conventions these variables are related to each other by

$$k = \frac{2\pi}{\lambda}, \qquad \omega = \frac{2\pi c}{\lambda} \tag{1}$$

and

$$c = 299792458\,\mathrm{m/s} \tag{2}$$

where variables in **bold** are vectors. Thus, for example, in Cartesian coordinates the electric field

$$\mathbf{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{3}$$

variable	name
$\mathbf{k}$	spatial angular frequency
$\mathbf{r}$	position vector
ω	angular frequency
c	speed of light
λ	wavelength
t	time
$\mathbf{E}$	electric field
δt	full width at half maximum in the time domain
$\delta\omega$	full width at half maximum in the angular frequency domain

Table 1: Description of variables.

2 Pulses of Light

The popular plane wave solution to Maxwell's equations is

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{4}$$

(i.e. monochromatic light with angular frequency ω and position vector $\mathbf{r}$). As mentioned in the introduction, we are most interested in the behavior of *pulses* of light. Pulses are characterized not by one frequency, but a continuum of frequencies

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\mathbf{E}}(\omega)\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}\, \mathrm{d}\omega \tag{5}$$

Here $\tilde{\mathbf{E}}(\omega)$ represents the *envelope* of the pulse in its angular frequency domain.

The particular formulation of Equation 5 is that of the Fourier transform. A Fourier transform enables the representation of an arbitrary pulse by a linear superposition of simple oscillatory functions - here mutually orthogonal complex exponential functions which satisfy the electromagnetic wave equation. The Fourier transform and its inverse are defined by

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}\omega = \mathscr{F}^{-}\big(\tilde{f}(\omega)\big) \tag{6}$$

and

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}t = \mathscr{F}^{+}\big(f(t)\big) \tag{7}$$

Where the convention $\mathscr{F}^{+}$ is the Fourier transform, and $\mathscr{F}^{-}$ is its inverse. In the same way, pulses can be represented in both the time and angular frequency domain as a Fourier transform pair

$$E(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{E}(\omega)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}\omega \tag{8}$$

$$\tilde{E}(\omega) = \int_{-\infty}^{\infty} E(t)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}t \tag{9}$$

$$\mathscr{F}^{+}\big(E(t)\big) = \tilde{E}(\omega) \tag{10}$$

$$\mathscr{F}^{-}\big(\tilde{E}(\omega)\big) = E(t) \tag{11}$$

3 Envelope Functions

The mathematical description of the pulse envelope is carried out with variables as designated in Figure 2. The envelopes are described in either the time or angular frequency domain by $f(t)$ or $\tilde{f}(\omega)$, respectively. The variables δt and $\delta\omega$ represent the *full width at half maximum* (FWHM) – the width of $|f(t)|^2$ or $|\tilde{f}(\omega)|^2$ at half of its maximum value. The product of the two, $\delta t\delta\omega$, is the *time bandwidth product.* As a consequence of the Fourier relationship, as the width of the pulse increases in the time domain, the spectral width decreases in the frequency domain and vice-versa. In terms of the actual electric field E, which contains a fast oscillating real and imaginary part, the function f is related by $|f|^2 = |E|^2$ (f has no oscillatory component). This is detailed in Figure 1.

For all pulse types we have normalized them so that the temporal integral

$$\int_{-\infty}^{\infty} \big|f(t)\big|^2\, \mathrm{d}t = 1 \tag{12}$$

and the integral over angular frequency

$$\int_{-\infty}^{\infty} \big|\tilde{f}(\omega)\big|^2\, \mathrm{d}\omega = 2\pi \tag{13}$$

For convince, this section treats envelope functions in a purely mathematical sense; the variables themselves dimensionless. Some common pulse shapes for modern femtosecond lasers are plotted in Figure 3. They are the Gaussian, the sech, and the Lorentzian. These specific pulse shapes are important in part because they are so called *transform limited*: for a given spectrum they represent the shortest pulse duration possible and the time-bandwidth product $\delta t\delta\omega$ is minimized for that pulse type.

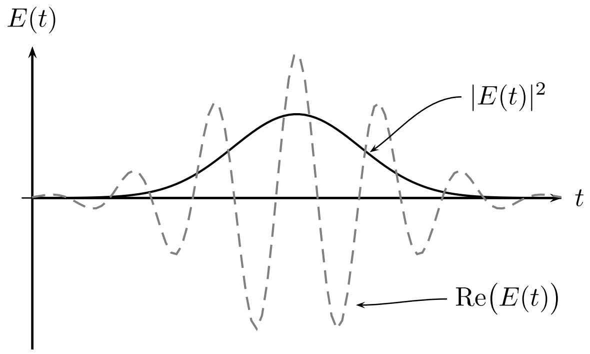

Figure 1: The envelope of a pulse is defined as the magnitude squared of its complex valued electric field. Since the function f in the derivations has no oscillatory component, the relation between f and E is given by $|f|^2 = |E|^2$.

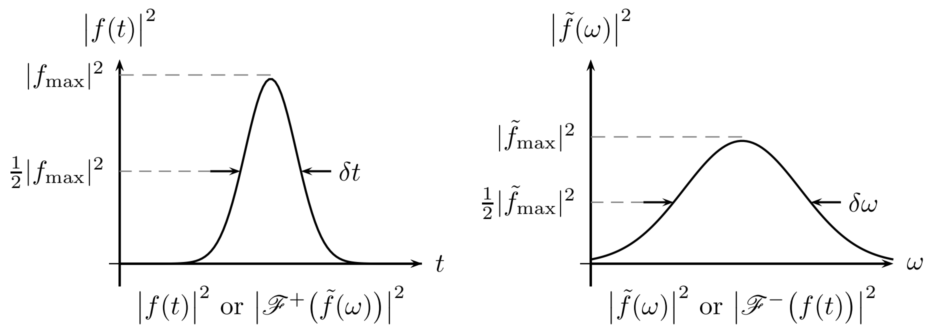

Figure 2: Definition of terms. A pulse can be represented in either the time $f(t)$ or angular frequency $\tilde{f}(\omega)$ domain through a Fourier transform relationship. The parameters δt or $\delta\omega$ are the width of the function at half its maximum value. As the width of the pulse increases in the time domain, the spectral width decreases in the frequency domain and vice-versa.

3.1 Gaussian Pulse

A Gaussian pulse is defined by the temporal envelope

$$f(t,\alpha) = A\,\mathrm{e}^{-t^2/\left(2\alpha^2\right)} \tag{14}$$

Where α is some, yet to be determined parameter (controlling the width of the pulse) and A is a normalization constant. The first step is to normalize Equation 14 and find A as per the condition described in Equation 12. To find do so, the well known Gaussian integral is computed

$$\int_{-\infty}^{\infty} \left| \mathrm{e}^{-t^2/\left(2\alpha^2\right)} \right|^2 \,\mathrm{d}t = A^2 \alpha \sqrt{\pi} \tag{15}$$

$$A = \sqrt{\frac{1}{\alpha\sqrt{\pi}}} \tag{16}$$

Substitution of A into Equation 14, the following is obtained

$$f(t,\alpha) = \left(\alpha\sqrt{\pi}\right)^{-1/2} \mathrm{e}^{-t^2/\left(2\alpha^2\right)} \tag{17}$$

In its current form using the variable α in effect controls the width of the envelope. Since we are interested in characterizing pulses by their FWHM, it is prudent to replace α with δt, such that δt represents the FWHM – the width of the magnitude squared envelope at half the maximum value.

$$|f(t,\alpha)|^2 = \frac{1}{2}\,|f(0)|^2 \tag{18}$$

$$\left| \mathrm{e}^{-t^2/\left(2\alpha^2\right)} \right|^2 = \frac{1}{2}\,\mathrm{e}^0 \tag{19}$$

$$\mathrm{e}^{-t^2/\left(2\alpha^2\right)} = \frac{1}{\sqrt{2}} \tag{20}$$

$$-\frac{t^2}{2\alpha^2} = \log\left(\frac{1}{\sqrt{2}}\right) \tag{21}$$

$$\frac{t^2}{2\alpha^2} = \frac{\log 2}{2} \tag{22}$$

$$t^2 = \alpha^2 \log 2 \tag{23}$$

$$t = \pm\alpha\sqrt{\log 2} \tag{24}$$

$$\delta t = 2\alpha\sqrt{\log 2} \quad , \quad \alpha = \frac{\delta t}{2\sqrt{\log 2}} \tag{25}$$

Note that the solution in terms of t is symmetric about $t = 0$. The parameter δt is then $2|t|$, fitting with the definition of the FWHM (Figure 2).

The variable α can then be inserted into Equation 17, obtaining our final expression.

$$\boxed{f(t,\delta t) = \left(\frac{2\sqrt{\log 2}}{\sqrt{\pi}\,\delta t}\right)^{1/2} \mathrm{e}^{-2t^2 \log 2/\delta t^2}} \tag{26}$$

We now have an expression where δt describes the full width at half maximum of the magnitude squared envelope. That is to say,

$$\left|f(t,\delta t)\right|^2 = \frac{1}{2}\left|f(0,\delta t)\right|^2 \tag{27}$$

The envelope is also normalized. By taking the Fourier transform of Equation 26, a complementary equation $\tilde{f}(\omega,\delta t)$, in terms of angular frequency (and the same parameter δt) can be obtained

$$\mathscr{F}^{+}\big(f(t,\delta t)\big) = \tilde{f}(\omega,\delta t) \tag{28}$$

$$= \int_{-\infty}^{\infty} \left(\frac{2\sqrt{\log 2}}{\sqrt{\pi}\,\delta t}\right)^{1/2} \mathrm{e}^{-2t^2 \log 2/\delta t^2}\, \mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t \tag{29}$$

From a table of indefinite integrals[2], an integral of the same form is found to be

$$\int_{-\infty}^{\infty} \mathrm{e}^{-at^2}\,\mathrm{e}^{-2bt}\,\mathrm{d}t = \sqrt{\frac{\pi}{a}}\,\mathrm{e}^{b^2/a} \tag{30}$$

Equating terms in the exponents from Equation 29 we obtain

$$a = \frac{2\log 2}{\delta t^2} \quad , \quad b = \frac{-i\omega}{2} \tag{31}$$

The integral can then be evaluated

$$\tilde{f}(\omega,\delta t) = \int_{-\infty}^{\infty} \left(\frac{2\sqrt{\log 2}}{\sqrt{\pi}\,\delta t}\right)^{1/2} \mathrm{e}^{-2t^2\log 2/\delta t^2}\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t \tag{32}$$

$$= \left(\frac{\pi\delta t^2}{2\log 2}\right)^{1/2}\left(\frac{2\sqrt{\log 2}}{\sqrt{\pi}\,\delta t}\right)^{1/2} \mathrm{e}^{-\omega^2\delta t^2/(8\log 2)} \tag{33}$$

therefore

$$\boxed{\tilde{f}(\omega,\delta t) = \left(\frac{\sqrt{\pi}\,\delta t}{\sqrt{\log 2}}\right)^{1/2} \mathrm{e}^{-\omega^2\delta t^2/(8\log 2)}} \tag{34}$$

Equation 34 is also normalized. Note that

$$\int_{-\infty}^{\infty} \left|\tilde{f}\,(\omega)\right|^2 = 2\pi \tag{35}$$

This is consistent with the Fourier transformed pulse defined in units of angular frequency.

At this point it is important to point out that the pulse envelope in both the time and angular frequency domain have been characterized entirely in terms of the (temporal) FWHM δt. This is not a mistake. The intention here is to simplify the (eventual) numerical computation by having one parameter controlling the width of the pulse. For example, if a 40 fs pulse was to be obtained by sampling in angular frequency space, the integral

$$f(t, 40\,\mathrm{fs}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega, 40\,\mathrm{fs})\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega \tag{36}$$

would be computed to obtain the pulse (in units of femtoseconds). Of course, the actual FWHM of the pulse $\tilde{f}(\omega,\delta t)$ is not δt. This will be addressed shortly.

The literature often quotes what is referred to as the *time bandwidth product*, $\delta t\delta\omega$. As written, Equation 26 has a FWHM of δt – the magnitude squared of the function decreases to half its maximum value for $t = \pm\delta t/2$. The FWHM of Equation 34 can be found in an analogous way by solving the following equation with width δt

$$\left|\tilde{f}(\omega,\delta t)\right|^2 = \frac{1}{2}\left|\tilde{f}(0,\delta t)\right|^2 \tag{37}$$

$$\left|\mathrm{e}^{-\omega^2\delta t^2/(8\log 2)}\right|^2 = \frac{1}{2}\,\mathrm{e}^0 \tag{38}$$

$$\mathrm{e}^{-\omega^2\delta t^2/(8\log 2)} = \frac{1}{\sqrt{2}} \tag{39}$$

$$\frac{\omega^2\delta t^2}{8\log 2} = \frac{\log 2}{2} \tag{40}$$

$$\omega^2 = \left(\frac{2\log 2}{\delta t}\right)^2 \tag{41}$$

$$\omega = \pm\frac{2\log 2}{\delta t} \tag{42}$$

$$\delta\omega = \frac{4\log 2}{\delta t} \tag{43}$$

Note in the last step we have defined $\delta\omega$ to be the FWHM of $\tilde{f}(\omega)$. Multiplying them together yields the time bandwidth product

$$\begin{aligned}\delta t \delta\omega &= \delta t \frac{4\log 2}{\delta t} &(44)\\ &= 4\log 2 \approx 2.7726 &(45)\end{aligned}$$

Because it is in terms of angular frequency, this result differs from the time bandwidth product, often quoted in literature in terms of ordinary frequency, $\delta t\delta\nu$, by a factor of $(2\pi)^{-1}$

$$\delta t\delta\nu = \frac{\delta t\delta\omega}{2\pi} = \frac{2\log 2}{\pi} \approx 0.4413 \tag{46}$$

Gaussian functions of this form have a useful expression for their definite integrals. This can be taken advantage of to determine what integration boundaries must be chosen to encompass a certain proportion of the total pulse energy.

For Gaussians symmetric about the origin, this integral is expressed in terms of the error function erf

$$\int_0^a \mathrm{e}^{-q^2t^2}\,\mathrm{d}t = \frac{\sqrt{\pi}}{2q}\,\mathrm{erf}(qa) \tag{47}$$

Because the Gaussian is symmetric about the origin $t = 0$, this can be trivially rewritten

$$\int_{-a}^a \mathrm{e}^{-q^2t^2}\,\mathrm{d}t = \frac{\sqrt{\pi}}{q}\,\mathrm{erf}(qa) \tag{48}$$

Using this form to evaluate Equation 26 with $q = 2\sqrt{\log 2}/\delta t$,

$$\begin{aligned}\int_{-a}^a |f(t,\delta t)|^2\,\mathrm{d}t &= \int_{-a}^a \frac{2\sqrt{\log 2}}{\sqrt{\pi}\delta t}\,\mathrm{e}^{-4t^2\log 2/\delta t^2}\,\mathrm{d}t &(49)\\ &= \frac{2\sqrt{\log 2}}{\sqrt{\pi}\delta t}\,\frac{\sqrt{\pi}\delta t}{2\sqrt{\log 2}}\,\mathrm{erf}\left(\frac{2a\sqrt{\log 2}}{\delta t}\right) &(50)\\ &= \mathrm{erf}\left(\frac{2a\sqrt{\log 2}}{\delta t}\right) &(51)\end{aligned}$$

Solving for a, we introduce a variable X such that X represents a proportion of the total pulse energy

$$\begin{aligned}\mathrm{erf}\left(\frac{2a\sqrt{\log 2}}{\delta t}\right) &= X &(52)\\ \frac{2a\sqrt{\log 2}}{\delta t} &= \mathrm{inverf}(X) &(53)\\ a &= \frac{\delta t}{2\sqrt{\log 2}}\,\mathrm{inverf}(X) &(54)\end{aligned}$$

Having found a, let $\zeta(\delta t, X)$ be defined as the function returning a,

$$\zeta(\delta t, X) = \frac{\delta t}{2\sqrt{\log 2}}\,\mathrm{inverf}(X) \tag{55}$$

Where $\mathrm{inverf}(X)$ is the inverse error function. This function expresses the bounds required to obtain a pulse with an energy of X of the total energy of an ideal Gaussian pulse. For example, the window for a 40 fs pulse encompassing 99 % of the total pulse energy would have bounds

$$\zeta(40\,\mathrm{fs}, 0.99) = \frac{40\,\mathrm{fs}}{2\sqrt{\log 2}}\,\mathrm{inverf}(0.99) \approx 43.75\,\mathrm{fs} \tag{56}$$

Therefore the window should have bounds at $\pm\zeta(40\,\mathrm{fs}, 0.99)$.

An equivalent set of boundaries for the Fourier transformed pulse can be found by the same procedure. Let $\tilde{\zeta}((\delta t, X)$ be defined as the analog to Equation 55: a function which returns the symmetric bounds in angular frequency space for which the integral of the magnitude squared envelope returns the proportion X of the total pulse energy.

$$\tilde{\zeta}(\delta t, X) = \frac{2\sqrt{\log 2}}{\delta t}\ \mathrm{inverf}(X) \tag{57}$$

3.2 Sech Pulse

In its simplest form, the sech pulse is defined

$$f(t, \alpha) = A\ \mathrm{sech}\left(\frac{2t}{\alpha}\right) \tag{58}$$

$$\tag{59}$$

To solve the integral

$$\int_{-\infty}^{\infty} |f(t, \alpha)|^2\,\mathrm{d}t \tag{60}$$

and obtain the normalizing constant A, we use an integral of the form[1]

$$\int \frac{\mathrm{d}x}{\cosh^n(ax)} = \frac{\sinh(ax)}{a\,(n-1)\cosh^{n-1}(ax)} + \frac{n-2}{n-1}\int \frac{\mathrm{d}x}{\cosh^{n-2}(ax)} \tag{61}$$

noting sech $x = \cosh^{-1} x$. With $n = 2$, and $a = 2/\alpha$ this becomes

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\cosh^2\left(\frac{2t}{\alpha}\right)} &= \left.\frac{\alpha\sinh\left(\frac{2t}{\alpha}\right)}{2\cosh\left(\frac{2t}{\alpha}\right)}\right|_{-\infty}^{\infty} + 0 && (62)\\
&= \left.\frac{\alpha}{2}\tanh\left(\frac{2t}{\alpha}\right)\right|_{-\infty}^{\infty} && (63)\\
&= \frac{\alpha}{2}\left(\lim_{t\to\infty}\tanh\left(\frac{2t}{\alpha}\right) - \lim_{t\to-\infty}\tanh\left(\frac{2t}{\alpha}\right)\right) && (64)
\end{aligned}$$

The second limit can be found by rewriting the hyperbolic tangent in terms of exponentials

$$\begin{aligned}
\tanh\left(\frac{2t}{\alpha}\right) &= \frac{\mathrm{e}^{2t/\alpha} - \mathrm{e}^{-2t/\alpha}}{\mathrm{e}^{2t/\alpha} + \mathrm{e}^{-2t/\alpha}} \cdot \frac{\mathrm{e}^{2t/\alpha}}{\mathrm{e}^{2t/\alpha}} && (65)\\
&= \frac{\mathrm{e}^{4t/\alpha} - 1}{\mathrm{e}^{4t/\alpha} + 1} && (66)\\
\lim_{t\to-\infty}\frac{\mathrm{e}^{4t/\alpha} - 1}{\mathrm{e}^{4t/\alpha} + 1} &= -1 && (67)
\end{aligned}$$

and the first through a similar argument but by multiplication with a different unity function

$$\begin{aligned}
\tanh\left(\frac{2t}{\alpha}\right) &= \frac{\mathrm{e}^{2t/\alpha} - \mathrm{e}^{-2t/\alpha}}{\mathrm{e}^{2t/\alpha} + \mathrm{e}^{-2t/\alpha}} \cdot \frac{\mathrm{e}^{-2t/\alpha}}{\mathrm{e}^{-2t/\alpha}} && (68)\\
&= \frac{1 - \mathrm{e}^{-4t/\alpha}}{1 + \mathrm{e}^{-4t/\alpha}} && (69)\\
\lim_{t\to\infty}\frac{1 - \mathrm{e}^{-4t/\alpha}}{1 + \mathrm{e}^{-4t/\alpha}} &= 1 && (70)
\end{aligned}$$

In terms of the original integral this evaluates to

$$\int_{-\infty}^{\infty} A \frac{\mathrm{d}t}{\cosh^2\left(\frac{2t}{\alpha}\right)} = 1 \tag{71}$$

$$A^2 \frac{\alpha}{2}\big(1-(-1)\big) = 1 \tag{72}$$

$$A = \frac{1}{\sqrt{\alpha}} \tag{73}$$

which is plugged into the original equation, normalizing it and obtaining

$$f(t,\alpha) = \frac{1}{\sqrt{\alpha}}\ \mathrm{sech}\left(\frac{2t}{\alpha}\right) \tag{74}$$

Solving for the parameter α

$$|f(t,\alpha)|^2 = \frac{1}{2}\,|f(0,\alpha)|^2 \tag{75}$$

$$\mathrm{sech}\left(\frac{2t}{\alpha}\right) = \frac{1}{\sqrt{2}} \tag{76}$$

$$\frac{2t}{\alpha} = \mathrm{arcsech}\left(\frac{1}{\sqrt{2}}\right) \tag{77}$$

$$t = \pm\frac{\alpha}{2}\mathrm{arcsech}\left(\frac{1}{\sqrt{2}}\right) \tag{78}$$

$$\delta t = \alpha\,\mathrm{arcsech}\left(\frac{1}{\sqrt{2}}\right) \quad , \quad \alpha = \frac{\delta t}{\mathrm{arcsech}\left(\frac{1}{\sqrt{2}}\right)} \tag{79}$$

using the identity

$$\mathrm{arcsech}(x) = \log\frac{1+\sqrt{1-x^2}}{x} \quad , \quad 0 < x \leq 1 \tag{80}$$

the function can be expressed in terms of exponentials. When α is substituted into Equation 74, the final form in the time domain is obtained

$$\boxed{f(t,\delta t) = \left(\frac{\log\big(1+\sqrt{2}\big)}{\delta t}\right)^{1/2} \mathrm{sech}\left(\frac{2t\log\big(1+\sqrt{2}\big)}{\delta t}\right)} \tag{81}$$

To take the Fourier transform of Equation 81, a table of integrals[2] is used to express the Fourier transform in terms of Gamma functions

$$\mathscr{F}^+\big(\mathrm{sech}^n\,(t/\alpha)\big) = \tilde{f}(\omega,\delta t) \tag{82}$$

$$= \frac{2^{n-1}}{\Gamma(n)}\,\Gamma\left(\frac{n+i\omega\alpha}{2}\right)\Gamma\left(\frac{n-i\omega\alpha}{2}\right) \tag{83}$$

along with the identity[1]

$$\Gamma\left(\frac{1}{2}+iy\right)\Gamma\left(\frac{1}{2}-iy\right) = \left|\,\Gamma\left(\frac{1}{2}+iy\right)\right|^2 \tag{84}$$

$$= \frac{\pi}{\cosh(\pi y)} \tag{85}$$

to obtain the Fourier transformed equation

$$\boxed{\tilde{f}(\omega,\delta t) = \pi\left(\frac{\delta t}{4\log\big(1+\sqrt{2}\big)}\right)^{1/2} \mathrm{sech}\left(\frac{\pi\omega\,\delta t}{4\log\big(1+\sqrt{2}\big)}\right)} \tag{86}$$

As in the case of the Gaussian pulse, the FWHM of the sech pulse in the time domain is the parameter δt. Solving for the width of the pulse in terms of $\tilde{f}(\omega, \delta t)$

$$\left|\tilde{f}(\omega, \delta t)\right|^2 = \frac{1}{2}\left|\tilde{f}(0, \delta t)\right|^2 \tag{87}$$

$$\operatorname{sech}\left(\frac{\pi\omega\,\delta t}{4\log\left(1+\sqrt{2}\right)}\right) = \frac{1}{\sqrt{2}} \tag{88}$$

$$\frac{\pi\omega\,\delta t}{4\log\left(1+\sqrt{2}\right)} = \operatorname{arcsech}\left(\frac{1}{\sqrt{2}}\right) \tag{89}$$

$$\omega = \pm\frac{4\log^2\left(1+\sqrt{2}\right)}{\pi\,\delta t} \tag{90}$$

$$\delta\omega = \frac{8\log^2\left(1+\sqrt{2}\right)}{\pi\,\delta t} \tag{91}$$

$$\tag{92}$$

and by multiplication of δt with $\delta\omega$ the time bandwidth product is obtained

$$\delta t\delta\omega = \delta t\,\frac{8\log^2\left(1+\sqrt{2}\right)}{\pi\,\delta t} \tag{93}$$

$$= \frac{8\log^2\left(1+\sqrt{2}\right)}{\pi} \approx 1.9782 \tag{94}$$

In terms of ordinary frequency

$$\delta t\delta\nu = \frac{\delta t\delta\omega}{2\pi} = \left(\frac{2\log\left(1+\sqrt{2}\right)}{\pi}\right)^2 \approx 0.3148 \tag{95}$$

The sech pulse also has a useful formula for calculating the total energy contained within a region. Beginning with the definite integral of Equation 63, and following the same procedure as was taken for the Gaussian pulse

$$\int_{-a}^{a}\operatorname{sech}^2\left(\frac{2t}{\alpha}\right) = \frac{\alpha}{2}\tanh\left(\frac{2t}{\alpha}\right)\Bigg|_{-a}^{+a} \tag{96}$$

$$= \frac{1}{2}\left(\tanh\left(\frac{2a\log\left(1+\sqrt{2}\right)}{\delta t}\right) - \tanh\left(\frac{-2a\log\left(1+\sqrt{2}\right)}{\delta t}\right)\right) \tag{97}$$

$$= \tanh\left(\frac{2a\log\left(1+\sqrt{2}\right)}{\delta t}\right) \tag{98}$$

then, solving a for a proportion of the pulse energy X the following is obtained

$$\tanh\left(\frac{2a\log\left(1+\sqrt{2}\right)}{\delta t}\right) = X \tag{99}$$

$$\frac{2a\log\left(1+\sqrt{2}\right)}{\delta t} = \operatorname{arctanh}(X) \tag{100}$$

$$a = \frac{\delta t}{2}\frac{\operatorname{arctanh}(X)}{\log\left(1+\sqrt{2}\right)} \tag{101}$$

And in the same regards as Equation 55,

$$\zeta(\delta t, X) = \frac{\delta t}{2}\frac{\operatorname{arctanh}(X)}{\log\left(1+\sqrt{2}\right)} \tag{102}$$

or, in terms of exponential functions

$$\zeta(\delta t, X) = \frac{\delta t}{4} \frac{\log\left(\frac{1+X}{1-X}\right)}{\log\left(1+\sqrt{2}\right)} \tag{103}$$

The same procedure can be used to obtain the parameter $\tilde{\zeta}(\delta t, X)$,

$$\tilde{\zeta}(\delta t, X) = \frac{4 \ \text{arctanh}(X) \ \log\left(1+\sqrt{2}\right)}{\pi \delta t} \tag{104}$$

$$= \frac{2 \log\left(\frac{1+x}{1-x}\right) \log\left(1+\sqrt{2}\right)}{\pi \delta t} \tag{105}$$

3.3 Lorentzian Pulses

Beginning in the time domain with a basic form of a Lorentzian envelope function,

$$f(t, \alpha) = \frac{A}{1+\alpha^2 t^2} \tag{106}$$

Solving for the normalizing coefficient A using a table of definite integrals[2]

$$\int_{-\infty}^{\infty} |f(t,\alpha)|^2 \, \mathrm{d}t = 1 \tag{107}$$

$$\frac{A^2 \pi}{2\alpha} = 1 \tag{108}$$

$$A = \sqrt{\frac{2\alpha}{\pi}} \tag{109}$$

and inserting it into Equation 106, the following is obtained

$$f(t, \alpha) = \sqrt{\frac{2\alpha}{\pi}} \frac{1}{1+\alpha^2 t^2} \tag{110}$$

from which α can be determined

$$|f(t,\alpha)|^2 = \frac{1}{2} |f(0,\alpha)|^2 \tag{111}$$

$$\frac{1}{1+\alpha^2 t^2} = \frac{1}{\sqrt{2}} \tag{112}$$

$$\sqrt{2} = 1 + \alpha^2 t^2 \tag{113}$$

$$\sqrt{2} - 1 = \alpha^2 t^2 \tag{114}$$

$$t^2 = \frac{\sqrt{2}-1}{\alpha^2} \tag{115}$$

$$t = \pm \left(\frac{\sqrt{2}-1}{\alpha^2} \right)^{1/2} \tag{116}$$

$$\delta t = 2\frac{\sqrt{\sqrt{2}-1}}{\alpha} \quad , \quad \alpha = 2\frac{\sqrt{\sqrt{2}-1}}{\delta t} \tag{117}$$

whereby substitution into Equation 110 yields the expression for the Lorentzian pulse in the time domain

$$\boxed{f(t, \delta t) = \left(\frac{4\sqrt{\sqrt{2}-1}}{\pi \delta t} \right)^{1/2} \frac{1}{1 + 4\left(\sqrt{2}-1\right) t^2/\delta t^2}} \tag{118}$$

The Fourier transform of Equation 118 is given by

$$\mathscr{F}\big(f(t,\delta t)\big) = \tilde{f}(\omega,\delta t) = \int_{-\infty}^{\infty} f(t,\delta t)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t \tag{119}$$

Using a similar integral form from the table[2]

$$\int_{-\infty}^{\infty} \frac{1}{1+at^2}\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t = \frac{\pi}{\sqrt{a}}\,\mathrm{e}^{-|\omega|/\sqrt{a}} \tag{120}$$

Substitution yields

$$\boxed{\tilde{f}(\omega,\delta t) = \left(\frac{\pi\delta t}{\sqrt{\sqrt{2}-1}}\right)^{1/2} \exp\left(\frac{-|\omega|\delta t}{2\sqrt{\sqrt{2}-1}}\right)} \tag{121}$$

Finally, determining the time bandwidth product of $\tilde{f}(\omega,\delta t)$,

$$\left|\tilde{f}(\omega,\delta t)\right|^2 = \frac{1}{2}\left|\tilde{f}(0,\delta t)\right|^2 \tag{122}$$

$$\exp\left(\frac{-|\omega|\delta t}{2\sqrt{\sqrt{2}-1}}\right) = \frac{1}{\sqrt{2}} \tag{123}$$

$$\frac{|\omega|\delta t}{2\sqrt{\sqrt{2}-1}} = \frac{\log 2}{2} \tag{124}$$

$$|\omega| = \frac{2\log 2\sqrt{\sqrt{2}-1}}{2\delta t} \tag{125}$$

$$\omega = \pm\frac{2\log 2\sqrt{\sqrt{2}-1}}{2\delta t} \tag{126}$$

$$\delta\omega = \frac{2\log 2\sqrt{\sqrt{2}-1}}{\delta t} \tag{127}$$

leaves

$$\delta t\delta\omega = 2\log 2\sqrt{\sqrt{2}-1} \approx 0.8922 \tag{128}$$

And in terms or ordinary frequency

$$\delta t\delta\nu = \frac{\delta t\delta\omega}{2\pi} = \frac{\log 2\sqrt{\sqrt{2}-1}}{\pi} \approx 0.142 \tag{129}$$

The Lorentzian pulse envelope in the time domain does not have a functional form for the proportional energy, as did the Gaussian and the sech pulses. It is therefore left to be evaluated numerically if needed by using the equation

$$\int_{-a}^{a} |f(t,\delta t)|^2\;\mathrm{d}t = \frac{2}{\pi}\left(\frac{a\alpha}{1+a^2\alpha^2} + \arctan(a\,\alpha)\right) = X \tag{130}$$

In terms of angular frequency however, a closed form can be found to be

$$\int_{-\tilde{a}}^{\tilde{a}} \left|\tilde{f}(\omega,\delta t)\right|^2\;\mathrm{d}\omega = 2\pi\left(1-\exp\left(-\sqrt{1+\sqrt{2}}\,a\,\delta t\right)\right) = 2\pi X \tag{131}$$

with a solution

$$\tilde{\zeta}(\delta t, X) = \frac{-\log(1-X)}{\delta t\sqrt{1+\sqrt{2}}} \tag{132}$$

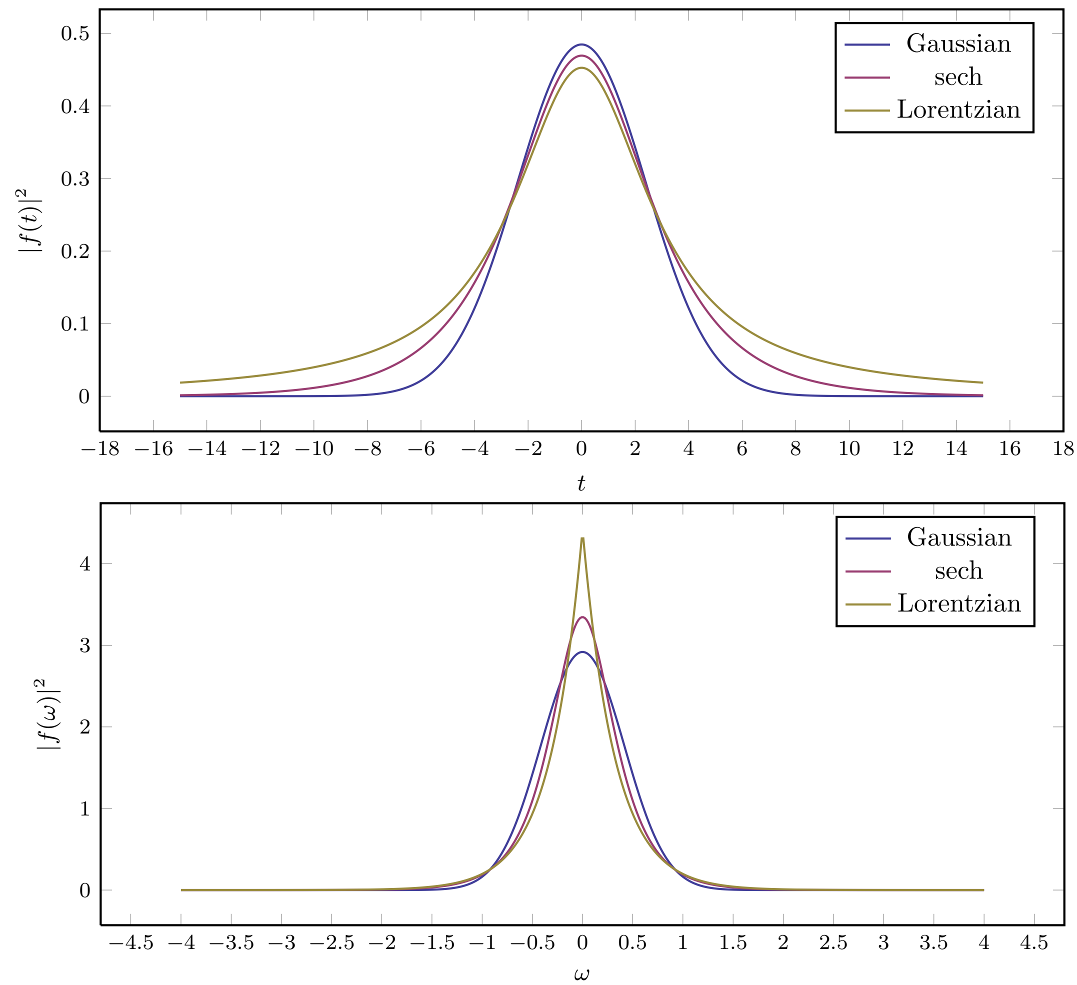

Figure 3: Plot of $|f(t)|^2$ and $|f(\omega)|^2$ for the Gaussian, sech, and Lorentzian pulse envelopes. Functions are normalized such that the total area under their curve is unity. Units are arbitrary ($\delta t = 4$).

4 Changes to this Document

- **14 October 2011**: Initial draft converted from thesis.
- **28 May 2012**: Cosmetic changes, new version of Figure 3.

References

[1] M. Abramowitz and I.A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. Dover publications, 1964.

[2] I.S. Gradshteyn, I.M. Ryzhik, A. Jeffrey, D. Zwillinger, and S. Technica. *Table of integrals, series, and products*, volume 1980. Academic press New York, 1965.

[3] Daniel A. Steck. *Classical and Modern Optics*. 2008.